# Extremal Graph Theory

Notes

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*„Fortunately, abacus wasn't in Hitlerjugend."*

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# 1. First extremal problems

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**Theorem 1** (Mantel, 1907). The maximum number of edges in a graph on $n$ vertices with no triangles in equal to $\left\lfloor \frac{n^2}{4} \right\rfloor$.

**Proof** (By induction). Lower bound: complete balanced bipartite graph.

Upper bound by induction. $n = 1, 2$ is fine. Now we assume $n > 2$. Let $G$ be a graph on $n$ vertices without triangles. Take $x, y \in V(G)$ such that $xy \in E(G)$. No other vertex is connected both to $x$ and $y$. We have $e(G) \leq e(G - \{x, y\}) + 1 + n - 2 \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + n - 1 = \left\lfloor \frac{n^2}{4} \right\rfloor$. □

**Proof** (By counting). If $G$ has no triangles, then for every edge $xy \in E(G)$ we have $\mathrm{d}(x) + \mathrm{d}(y) \leq n$. Summing over all edges $\sum_{x \in V(G)} \mathrm{d}(x)^2 = \sum_{xy \in E(G)} \mathrm{d}(x) + \mathrm{d}(y) \leq n \cdot e(G)$. Using Cauchy-Schwarz we get $4e(G)^2 = \left(\sum_{x \in V(G)} \mathrm{d}(x)\right)^2 \leq n \cdot \sum_{x \in V(G)} \mathrm{d}(x)^2$. Thus $\frac{4e(G)^2}{n} \leq ne(G) \implies e(G) \leq \frac{n^2}{4}$. □

**Theorem 2** (Turán, 1941). The number of edges in a graph on $n$ vertices with no $K_{k+1}$ is at most $\left(1 - \frac{1}{k}\right)\frac{n^2}{2}$.

**Proof.** Lower bound: complete $k$-partite balanced graph also known as the Turán graph $T_k(n)$.

Upper bound by induction. For $n \leq k$ it is fine. For $n > k$ and $G$ without $K_{k+1}$ we can assume $G$ has $K_k$ (if it doesn't, then we can add edges until it does). Let $A$ be a clique of order $k$ in $G$. $e(G) \leq e(G - A) + e(G[A]) + (n - k)(k - 1) \leq \left(1 - \frac{1}{k}\right)\frac{(n-k)^2}{2} + \frac{k(k-1)}{2} + (n - k)(k - 1) = \left(1 - \frac{1}{k}\right)\frac{n^2}{2}$. □

# 2. Turán theorem

2026-10-13

**Definition 3.** The extremal number (Turán number) of a graph $F$, denoted $\mathrm{ex}(n, F)$, is the maximum number of edges in an $n$-vertex graph without $F$ as a subgraph (so an $F$-free graph).

**Example.** We know that $\mathrm{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$ and $\left(1 - \frac{1}{k}\right)\frac{n^2}{2} - O(n) \leq \mathrm{ex}(n, K_{k+1}) \leq \left(1 - \frac{1}{k}\right)\frac{n^2}{2}$.

**Theorem 4** (Turán, 1946). $\mathrm{ex}(n, K_{k+1}) = e(T_k(n))$. Moreover, $T_k(n)$ is the unique extremal graph.

**Proof** (Erdős, 1970). We proceed by induction. $k = 1$ is easy. Let $G$ be an extremal graph for $k > 1$. Take a vertex $x$ of maximum degree. Let $S$ be the neighbourhood of $x$ and let $T = V(G) \setminus S$. Let $H$ be a graph on $V(G)$ such that $H[S] = G[S]$, $H$ contains all edges between $S$ and $T$ and no edges inside $T$. $H$ does not contain $K_{k+1}$ as a subgraph. We have

$$e(G) = e(G[S]) + \sum_{v \in T} \mathrm{d}(v) - e(G[T]) \leq e(H[S]) + |T||S| - e(G[T]) = e(H) - e(G[T]).$$

As $G$ is extremal, we have $e(G[T]) = 0$. $G[S]$ does not contain $K_k$ and has the maximum number of edges, so by induction it is complete multipartite. Thus, $G$ is also complete multipartite. For $G$ to be extremal, it needs to be balanced as well. □

**Proof** (Zykov, 1949). Let $G$ be an extremal graph. Suppose it is not complete multipartite. Then $G$ contains $x, y, z$ such that $xz \in E(G)$ and $xy, yz \notin E(G)$. If $\mathrm{d}(x) > \mathrm{d}(y)$, then remove $y$ and add a copy of $x$. We don't have a $K_{k+1}$ and there are more edges – a contradiction with extremality of $G$. Thus, we have $\mathrm{d}(x) \leq \mathrm{d}(y)$ and similarly $\mathrm{d}(z) \leq \mathrm{d}(y)$. Now remove $x$ and $z$, add two copies of $y$. There is no $K_{k+1}$ and there are more edges – a contradiction. □

**Theorem 5** (Katona, 1969). Let $u$ and $v$ be vectors taken independently, uniformly at random from some finite set of vectors in $\mathbb{R}^d$. Then $\mathbb{P}(\|u + v\| \geq x) \geq \frac{1}{2}\mathbb{P}(\|u\| \geq x)^2$ for any $x$.

**Proof.** Say that $N$ is the number of vectors and $n$ is the number of vectors of length at least $x$. We have $\mathbb{P}(\|u\| \geq x) = \frac{n}{N}$. We define a graph $G$ such that $V(G)$ are the vectors of length at least $x$ and $uv \in E(G) \iff \|u + v\| < x$. $G$ does not have triangles, as otherwise we have

$\|a+b+c\|^2 = \|a+b\|^2 + \|a+c\|^2 + \|b+c\|^2 - \|a\|^2 - \|b\|^2 - \|c\|^2 < 0$ for some vectors $a, b, c$. Thus, $e(G) \leq \frac{n^2}{4}$. We have $\mathbb{P}(\|u+v\| \geq x) \geq \frac{n^2 - 2\cdot\frac{n^2}{4}}{N^2} = \frac{1}{2}\left(\frac{n}{N}\right)^2$. □

**Definition 6.** Let $H$ be a graph and $t$ a positive integer. By $H[t]$ we denote the $t$-blow-up of $H$, which is a graph obtained from $H$ by replacing each vertex with an independent set of size $t$.

**Theorem 7** (Erdős, Stone, 1946)**.** For any integers $k, t \geq 1$ and $\varepsilon > 0$ there is $n_0$ such that for any $n \geq n_0$ we have $\mathrm{ex}(n, K_{k+1}[t]) \leq \left(1 - \frac{1}{k} + \varepsilon\right)\frac{n^2}{2}$.

**Theorem 8** (Erdős, Simonovits, 1966)**.** If $F$ is a non-empty graph, then

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F)-1}\right)\frac{n^2}{2} + o(n^2).$$

**Proof.** Let $k = \chi(F) - 1$. $T_k(n)$ certifies the lower bound, as it is $k$-colourable. Now we need to show that for any $\varepsilon > 0$, there is $n_0$ such that for $n \geq n_0$ we have $\mathrm{ex}(n, F) \leq \left(1 - \frac{1}{k} + \varepsilon\right)\frac{n^2}{2}$. Let $t = |V(F)|$ and consider $K_{k+1}[t]$. It contains $F$ (we can colour $F$ and using this colouring assign vertices to parts of $K_{k+1}[t]$). We conclude by Theorem 7. □

**Definition 9.** An edge $e$ in a graph $F$ is colour critical when $\chi(F - e) < \chi(F)$.

**Theorem 10** (Critical edge theorem, Simonovits, 1968)**.** Let $F$ be a graph having $\chi(F) = k+1$. For $n$ large enough, the unique extremal graph for $F$ is $T_k(n)$ if and only if $F$ has a colour critical edge.

**Proof.** ( $\Longrightarrow$ ) We find a copy of $F$ in $T_k(n)$ with one more edge, this edge is colour critical for $F$.

( $\Longleftarrow$ )

***Claim* 10.1.** Consider $t = 2|V(F)|$ and an $F$-free graph formed by $K_k[t]$ and an extra vertex $x$. $\mathrm{d}(x) \leq (k-1)t$ with equality when $x$ is connected to all vertices of $K_k[t]$ except for one part.

*Proof.* If $x$ is connected to less than $\frac{t}{2}$ vertices in two parts, then $\mathrm{d}(x) < (k-1)t$. From now on we assume $x$ has at least $\frac{t}{2}$ neighbours in all parts except one part $P$. If $x$ has at least one neighbour in $P$, then the graph has an $F$ – we connect $x$ to $P$ by the colour critical edge, we assign the rest using the chromatic number. Therefore $x$ has no neighbours in $P$ and $\mathrm{d}(x) \leq (k-1)t$ with equality only when $x$ is connected to all vertices of $K_k[t]$ but those in $P$. ◁

Let $G_n$ be an extremal graph for $F$ on $n$ vertices. $F$ is not contained in $T_k(n)$, so $e(G_n) \geq e(T_k(n))$. By Theorem 7, for large enough $n$ $G_n$ contains $K_k[t]$ for $t = 2|V(F)|$. Fix such a copy $K$ nad let $S$ be the set of the remaining $n - kt$ vertices of $G_n$. Let $e(S, K)$ denote the number of edges between $K$ and $S$. $G_n$ has no edges inside the parts of $K$, as otherwise $G_n$ contains $F$ (map this edge to the colour critical edge and the rest according to colour classes). By Claim 10.1 $e(S, K) \leq (n - kt)(k-1)t$. Thus,

$$e(G_n) = e(K_k[t]) + e(S, K) + e(G_n[S]) \leq e(K_k[t]) + (n - kt)(k-1)t + e(G_{n-kt}).$$

$T_k(n)$ contains a $K_k[t]$ and $T_k(n) - K_k[t]$ is simply $T_k(n - kt)$, so

$$e(T_k(n)) = e(K_k[t]) + (n - kt)(k-1)t + e(T_k(n - kt)).$$

Combining these inequalities we get $e(G_n) - e(T_k(n)) \leq e(G_{n-kt}) - e(T_k(n - kt))$. For each $i \in \{0, \ldots, kt-1\}$ the sequence $e(G_{nkt+i}) - e(T_k(nkt + i))$ is non-negative and non-increasing (when $n$ is large enough). Thus, it is convergent and eventually constant (as it is an integer sequence). This means that for large enough $n$ the previous inequality is an equality and $e(S, K) = (n - kt)(k-1)t$. By Claim 10.1 each vertex of $S$ is adjacent to all vertices of $K$ except those in one part. Colour each vertex of $K$ by its part and each vertex of $S$ by the part of $K$ not adjacent to it. This colouring is proper, as if two vertices of $S$ of the same colour are adjacent, then we can map the colour critical edge of $F$ to them and the rest to the vertices of $K$ by the colour classes. Thus, $G_n$ is $k$-partite. By maximality $G_n = T_k(n)$. □

**Corollary.** For $n$ large enough, the unique extremal graph for the odd cycle $C_{2k+1}$ is $T_2(n)$. In particular, $\mathrm{ex}(n, C_{2k+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor$.

## 3. Stability

2026-10-20

**Idea.** Let us examine the structure of almost extremal graphs. It often turns out that when the number of edges is close to extremal, then the graph itself resembles an extremal example. This concept is called stability.

**Theorem 11** (Füredi, 2015)**.** Let $k \geq 1$ and let $G$ be a $K_{k+1}$-free graph on $n$ vertices with $e(G) = e(T_k(n)) - t$. Then $G$ contains a $k$-partite subgraph $G'$ with $e(G') \geq e(G) - t$.

**Proof.** We build a partition of $G$. We start by taking $x_1$ as a vertex with maximal degree. Let $N_1$ be its neighbourhood and $V_1 = V(G) \setminus N_1$. Now suppose we already defined $x_j, V_j, N_j$ for $j = 1, \ldots, i-1$. Let $x_i$ be a vertex with maximal neighbourhood in $G[N_{i-1}]$. Call this neighbourhood $N_i$ and let $V_i = N_{i-1} \setminus N_i$. We have $x_i \in N_{i-1} \subseteq \cdots \subseteq N_1$ and the vertices $x_1, x_2, \ldots$ form a clique. Thus, this process stops after at most $k$ steps. For simplicity, we assume it stops after choosing $x_k$, that is, $x_k$ has no neighbours is $N_{k-1}$. Thus, $V_k = N_{k-1}$ and $V_1, \ldots, V_k$ is a partition of $V(G)$.

$N_i = V_{i+1} \cup \cdots \cup V_k$, so $\sum_{i=1}^{k} |V_i||N_i|$ is the number of edges in the complete $k$-partite graph with parts $V_1, \ldots, V_k$. This number cannot be bigger than the number of edges in a $k$-partite Turán graph. Thus, $\sum_{i=1}^{k} |V_i||N_i| \leq e(T_k(n)) = e(G) + t$. On the other hand, a vertex $x \in V_i$ has at most $|N_i|$ neighbours in $N_{i-1} = V_i \cup \cdots \cup V_k$ (by the choice of $x_i$). Summing up over all such vertices we count the edges inside every $V_i$ twice, so $e(G) + \sum_{i=1}^{k} e(G[V_i]) \leq \sum_{i=1}^{k} |V_i||N_i|$. This implies there are at most $t$ edges inside the sets $V_i$. Removing them gives us the desired $G'$. □

**Theorem 12** (Erdős-Simonovits stability theorem, 1966)**.** For any $\varepsilon > 0$ and graph $F$ with $\chi(F) = k + 1$ there exist $\delta > 0$ and $n_0$ such that if $G$ is an $F$-free $n$-vertex graph with $n \geq n_0$ and $e(G) \geq \left(1 - \frac{1}{k}\right)\frac{n^2}{2} - \delta n^2$, then $G$ can be obtained from $T_k(n)$ by adding and removing at most $\varepsilon n^2$ edges in total.

**Notation.** For $A \subseteq V(G)$ let $\mathrm{d}(A)$ denote the number of common neighbours of vertices in $A$.

**Theorem 13** (Asymptotic structure theorem)**.** If $G$ is an $n$-vertex extremal graph for a graph $F$ with $\chi(F) = k + 1$, then $\delta(G) = \left(1 - \frac{1}{k}\right)n + o(n)$.

**Proof.** As $T_k(n)$ if $F$-free, we have $e(G) \geq e(T_k(n)) = \left(1 - \frac{1}{k}\right)\frac{n^2}{2} - O(n)$. By Theorem 12, we know that $T_k(n)$ can be transformed into $G$ by adding and removing $o(n^2)$ edges. For a vertex $v$, let $r(v)$ denote the number of edges incident to $v$ removed in this process. Let $V_1, \ldots, V_k$ be the parts of $T_k(n)$ and let $V_1$ have minimal size ($|V_1| = \left\lfloor \frac{n}{k} \right\rfloor$). We have $\sum_{c \in V_1} r(v) = o(n^2)$. Let $S \subseteq V_1$ be such that $|S| = |V(F)|$ and $\sum_{v \in S} r(v)$ is minimal. By taking averages we have $\sum_{v \in S} r(v) \leq \frac{|S|}{|V_1|} \sum_{v \in V_1} r(v) = o(n)$.

Let $x \in V(G)$. We claim that $d(x) \geq \mathrm{d}(S) - 1$. If $x \in S$, then clearly $\mathrm{d}(x) \geq \mathrm{d}(S)$. Now assume $x \notin S$ and $\mathrm{d}(x) < \mathrm{d}(S) - 1$. Remove all edges incident to $x$ and connect $x$ to all common neighbours of $S$ (except, possibly, $x$). The graph now has more edges and remains $F$-free (if there is now a copy containing $x$, then by $|S| = |V(F)|$ we can replace $x$ by some vertex from $S$). This is a contradiction.

In $T_k(n)$ all $n - \left\lfloor \frac{n}{k} \right\rfloor$ vertices outside $V_1$ are common neighbours of $S$. Each removed edge destroys at most one of them. Thus,

$$\mathrm{d}(x) \geq \mathrm{d}(S) - 1 \geq n - \left\lfloor \frac{n}{k} \right\rfloor - \sum_{v \in S} r(v) - 1 \geq \left(1 - \frac{1}{k}\right)n - o(n).$$

On the other hand, by Theorem 8,

$$\delta(G) \leq \frac{2e(G)}{n} = \frac{2\,\mathrm{ex}(n,F)}{n} = \left(1-\frac{1}{k}\right)n + o(n).$$

□

**Theorem 14** (Andrásfai, Erdős, Sós, 1974). Every $n$-vertex $K_{k+1}$-free graph $G$ that satisfies $\delta(G) > \left(1-\frac{3}{3k-1}\right)n$ is $k$-partite.

**Proof** (Reiher, 2012). Take a counterexample with the biggest number of edges. Adding an edge always creates $K_{k+1}$.

$G$ is not complete multipartite. Thus, $G$ contains vertices $x, y, z$ such that $yz \in E(G)$ and $xy, xz \notin E(G)$. Adding $xy$ as an edge creates a clique, to there is a set of $k-1$ vertices that form a clique that is fully connected to both $x$ and $y$. The same is true for $xz$. Let $B$ be the intersection of these two cliques. Let $A$ and $C$ be the remaining parts of the cliques. Denote $|A| = a$, $|B| = b$, and $|C| = c$. Among all such configurations, take one with the largest $b$. We have $b \leq k-2$, otherwise we can add $y$ and $z$ and get $K_{k+1}$.

Assign weights to vertices: $A \cup C \cup \{x, y, z\}$ gets 1, $B$ gets 3, the rest gets 0. The total weight is $a + 3b + c + 3 = 2k + b + 1 \leq 3k - 1$. We have

$$\sum_{uv \in E(G)} w(u) + w(v) = \sum_{u \in V(G)} w(u)\mathrm{d}(u) > \left(1-\frac{3}{3k-1}\right)n(2k+b+1) \geq$$
$$\left(1-\frac{3}{2k+b+1}\right)(2k+b+1)n = (2k+b-2)n.$$

This means $\sum_{u \in V(G)} \sum_{v \in N(u)} w(v) > (2k+b-2)n$ and there exists $u$ such that $\sum_{v \in N(u)} w(v) \geq 2k+b-1$. This means $u$ is connected to everything in $B$ and all but at most 2 vertices of the rest.

If $u$ is connected to all of $A$, then $A \cup B \cup \{u\}$ is a $K_k$, so $u$ is not connected to $x$ nor $y$, as otherwise we have $K_{k+1}$. Thus, $u$ is connected to $z$ and all of $C$, and $C \cup B \cup \{z, u\}$ is a $K_{k+1}$. This means $u$ is not connected to all of $A$. Similarly $u$ is not connected to all of $C$.

Thus, $u$ is connected with $x, y, z, A \setminus \{\alpha\}, C \setminus \{\gamma\}$. Take $A' = A \setminus \{\alpha\}$, $B' = B \cup \{u\}$, $C' = C \setminus \{\gamma\}$. This creates a configuration with bigger $b$ and gives a contradiction. □

# 4. Supersaturation

2026-10-27

**Notation.** Let $N_s$ denote the number of cliques of order $s$ (in some graph known from context).

**Theorem 15** (Moon, Moser, 1962). For every $s \geq 2$ and an $n$-vertex graph $N_{s+1} \geq \frac{N_s}{s^2-1}\left(\frac{s^2 N_s}{N_{s-1}} - n\right)$.

**Proof.** Count triples $(S', x, y)$, where $S'$ is a clique of order $s-1$, $S' \cup \{x\}$ forms a $K_s$ and $S' \cup \{y\}$ forms a $K_s$. The number is

$$\sum_{S'} \mathrm{d}(S')^2 \geq \frac{1}{N_{s-1}}\left(\sum_{S'} \mathrm{d}(S')\right)^2 = \frac{1}{N_{s-1}}(sN_s)^2.$$

We have $N_{s+1}(s+1)s$ triples such that $x, y$ are connected (and distinct). If $x, y$ are not connected (or the same), then we have at most $\sum_{S:\text{clique of size } s} n - \mathrm{d}(S) = nN_s - (s+1)N_{s+1}$.

$$\frac{1}{N_{s-1}}s^2N_s^2 \leq (s+1)sN_{s+1} + nN_s - (s+1)N_{s+1} \implies N_{s+1} \geq \frac{N_s}{s^2-1}\left(\frac{s^2N_s}{N_{s-1}} - n\right).$$

□

**Corollary.** If $G$ is an $n$-vertex graph with $e(G) = \left(1 - \frac{1}{x}\right)\frac{n^2}{2}$, then for every $s \geq 1$ we have $N_{s+1} \geq \left(1 - \frac{s}{x}\right)\frac{n}{s+1}N_s$.

**Proof.** Induction on $s$. $s = 1$ is the assumption on the number of edges. Now

$$N_{s+1} \geq \frac{N_s}{s^2-1}\left(\frac{s^2N_s}{N_{s-1}} - n\right) \geq \frac{N_s}{s^2-1}\left(\frac{s^2\left(1 - \frac{s-1}{x}\right)\frac{n}{s}N_{s-1}}{N_{s-1}} - n\right) = \frac{N_s n}{s+1}\left(1 - \frac{s}{x}\right).$$

□

**Corollary.** For any $\varepsilon > 0$ and integer $s \geq 1$ there exists $c > 0$ such that any $n$-vertex graph with at least $\left(1 - \frac{1}{s} + \varepsilon\right)\frac{n^2}{2}$ edges has at least $cn^{s+1}$ copies of $K_{s+1}$.

**Proof.** Let $x$ be such that $e(G) = \left(1 - \frac{1}{x}\right)\frac{n^2}{2}$. This means that $\frac{1}{x} \leq \frac{1}{s} - \varepsilon$, which implies $x - s \geq \varepsilon x s > 0$. Then

$$N_{s+1} \geq \left(1 - \frac{s}{x}\right)\frac{n}{s+1}N_s \geq \prod_{i=1}^{s}\left(1 - \frac{i}{x}\right)\frac{n}{i+1}N_1 = \frac{n^{s+1}}{x^s(s+1)!}\prod_{i=1}^{s}(x-i) \geq$$
$$\frac{n^{s+1}}{x^s(s+1)!}(x-s)^s \geq \frac{\varepsilon^s s^s}{(s+1)!}n^{s+1}.$$

□

**Lemma 16.** For any real $q \geq 0$, $\varepsilon > 0$, and integer $m \geq 2$ there exists $\delta > 0$ such that every $n$-vertex graph $G$ with $e(G) \geq (q+\varepsilon)\binom{n}{2}$ contains at least $\delta\binom{n}{m}$ induced subgraphs of order $m$ with at least $\left(q + \frac{\varepsilon}{2}\right)\binom{m}{2}$ edges.

**Proof.** Each edge of $G$ is contained in $\binom{n-2}{m-2}$ sets of vertices of size $m$, so

$$\sum_{S \in \binom{V(G)}{m}} e(G[S]) = e(G)\binom{n-2}{m-2} \geq \binom{n-2}{m-2}(q+\varepsilon)\binom{n}{2}.$$

Let $\delta = \frac{\varepsilon}{4}$ and suppose the lemma is false. This means there are less than $\delta\binom{n}{m}$ sets with at least $\left(q + \frac{\varepsilon}{2}\right)\binom{m}{2}$ edges. Thus

$$\sum_{S \in \binom{V(G)}{m}} e(G[S]) < \delta\binom{n}{m}\binom{m}{2} + (1-\delta)\binom{n}{m}\left(q + \frac{\varepsilon}{2}\right)\binom{m}{2} = \binom{n}{m}\binom{m}{2}\left(\delta + (1-\delta)\left(q + \frac{\varepsilon}{2}\right)\right) =$$
$$\binom{n}{m}\binom{m}{2}\left(q + \frac{\varepsilon}{2} + \delta\left(1 - q - \frac{\varepsilon}{2}\right)\right) < \binom{n}{m}\binom{m}{2}\left(q + \frac{3}{4}\varepsilon\right).$$

This implies

$$\binom{n-2}{m-2}\binom{n}{2}(q+\varepsilon) < \binom{n}{m}\binom{m}{2}\left(q + \frac{3}{4}\varepsilon\right)$$

so $q + \varepsilon < q + \frac{3}{4}\varepsilon$, as $\binom{n-2}{m-2}\binom{n}{2} = \binom{n}{m}\binom{m}{2}$. A contradiction. □

**Theorem 17** (Erdős, Simonovits, 1983). For any graph $F$ and $\varepsilon > 0$ there exists $c > 0$ such that if $G$ is an $n$-vertex graph with $e(G) \geq \mathrm{ex}(n, F) + \varepsilon n^2$, then $G$ contains at least $cn^{|V(F)|}$ copies of $F$.

**Proof.** Take $q = 1 - \frac{1}{\chi(F)-1}$. Take $m$ large enough so that $\mathrm{ex}(m, F) < \left(q + \frac{\varepsilon}{2}\right)\binom{m}{2}$ by Theorem 8. For small $n$ we can take $c$ such that $cn^{|V(F)|} \leq 1$. For large $n$ we have $e(G) \geq (q+\varepsilon)\binom{n}{2}$ by Theorem 8. We apply Lemma 16. There are $\delta\binom{n}{m}$ subsets with at least $\left(q + \frac{\varepsilon}{2}\right)\binom{m}{2}$ edges (so containing $F$). Let $k = |V(F)|$. A copy of $F$ can appear in at most $\binom{n-k}{m-k}$ subsets. Thus, the number of copies of $F$ is at least

$$\frac{\delta\binom{n}{m}}{\binom{n-k}{m-k}} = cn^k$$

for some $c$. □

**Definition 18.** Let $g_s(n,e) = \min\{N_s(G) : G \text{ is an } n\text{-vertex graph with } e \text{ edges}\}$. For a given $\lambda \in (0,1)$ take any integer function $e(n)$ such that $0 \le e(n) \le \binom{n}{2}$ and $\lim_{n\to\infty} \frac{e(n)}{\binom{n}{2}} = \lambda$. Let $g_s(\lambda) = \lim_{n\to\infty} \frac{g_s(n,e(n))}{\binom{n}{s}}$. $g_s(\lambda)\binom{n}{s} + o(n^2)$ is the minimum number of $s$-cliques in graphs with $\lambda\binom{n}{2} + o(n^2)$ edges.

**Remark.** We know the bound $g_3(n,e) \ge \frac{e(4e-n^2)}{3n}$. This bound is tight for any $\lambda = \frac{k-1}{k}$. For the values in between, the function $g_3(\lambda)$ is given by a concave arc. This was proved using flag algebras.

# 5. Probabilistic method

2026-11-03

**Theorem 19.** Every graph $G$ contains a bipartite subgraph with at least $\frac{e(G)}{2}$ edges.

**Proof.** We assign vertices to two parts independently, uniformly at random. We then remove all edges in one part. An edge survives with probability $\frac{1}{2}$. □

**Theorem 20** (Erdős, 1947)**.** Let $R(k,\ell)$ denote the standard Ramsey number (2-colouring of edges). For any $n,k$ if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. In particular, $R(k,k) > \left(\frac{1}{e\sqrt{2}} + o(1)\right)k2^{\frac{k}{2}}$.

**Proof.** Colour edges independently, uniformly at random. $S$ of size $k$ induces a monochromatic clique with probability $2^{1-\binom{k}{2}}$. The statement follows by probabilistic method. □

**Theorem 21.** For any $n,k$ we have $R(k,k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$. In particular, $R(k,k) > \left(\frac{1}{e} + o(1)\right)k2^{\frac{k}{2}}$.

**Proof.** Colour edges independently, uniformly at random. If a set induces a monochromatic $K_k$, remove one of its vertices. Expected number of monochromatic cliques is $\binom{n}{k}2^{1-\binom{k}{2}}$. Thus, there is a colouring for which we remove at most this number of vertices. □

**Definition 22.** We say an event $A$ is independent from events $A_1,\dots,A_k$ if $A$ is independent from every event of the form $B_1 \cap \dots \cap B_k$, where each $B_i$ is $A_i$ or $\overline{A_i}$.

**Definition 23.** $D$ is a dependency graph for events $A_1,\dots,A_k$ if $V(G) = \{1,\dots,k\}$ and for every $i$ the event $A_i$ is independent from the family of all events $A_j$ such that $j \notin N[i]$.

**Theorem 24** (Lovász Local Lemma, symmetric version)**.** Let $A_1,\dots,A_k$ be events such that for each $i$ we have $\mathbb{P}(A_i) \le p$ and each $A_i$ is independent from all but at most $d$ other events. If $ep(d+1) \le 1$, then $\mathbb{P}\left(\bigcap_{i=1}^k \overline{A_i}\right) > 0$.

**Theorem 25** (Spencer, 1977)**.** For any $n,k$ if $\left(\binom{k}{2}\binom{n}{k-2} + 1\right)2^{1-\binom{k}{2}}e \le 1$, then $R(k,k) > n$. In particular, $R(k,k) > \left(\frac{\sqrt{2}}{e} + o(1)\right)k2^{\frac{k}{2}}$.

**Proof.** Colour edges independently, uniformly at random. Probability $S$ of size $k$ is monochromatic is $2^{1-\binom{k}{2}}$. Two sets $S, S'$ share an edge (and are not independent) when $|S \cap S'| \ge 2$. For each $S$, there are at most $\binom{k}{2}\binom{n}{k-2}$ choices of such $S'$. We conclude by Lovász Local Lemma. □

**Theorem 26** (Asymmetric Lovász Local Lemma; Erdős, Lovász, 1975)**.** Consider a dependency graph $D$ for some events $A_1,\dots,A_n$. If real numbers $x_1,\dots,x_n \in [0,1)$ satisfy $\mathbb{P}(A_i) \le x_i \prod_{j\in N(i)}(1-x_j)$ for all $i$, then $\mathbb{P}\left(\bigcap_{i=1}^n \overline{A_i}\right) \ge \prod_{i=1}^n (1-x_i)$.

**Proof.** We first prove by induction on $s$ that for every set $S \subseteq \{1,\dots,n\}$ with $|S| = s$ and for every $i$ we have $\mathbb{P}\left(A_i \mid \bigcap_{j\in S} \overline{A_j}\right) \le x_i$. $s = 0$ follows by one of the assumptions. Assume the statement

holds for every $s' < s$. Let $S_1 = \{j \in S : ij \in E(D)\}$ and $S_2 = S \setminus S_1$. We have

$$\mathbb{P}\left(A_i \mid \bigcap_{j\in S} \overline{A_j}\right) = \frac{\mathbb{P}\left(A_i \cap \left(\bigcap_{j\in S_1} \overline{A_j}\right) \mid \bigcap_{j\in S_2} \overline{A_j}\right)}{\mathbb{P}\left(\bigcap_{j\in S_1} \overline{A_j} \mid \bigcap_{j\in S_2} \overline{A_j}\right)} \leq \frac{\mathbb{P}\left(A_i \mid \bigcap_{j\in S_2} \overline{A_j}\right)}{\mathbb{P}\left(\bigcap_{j\in S_1} \overline{A_j} \mid \bigcap_{j\in S_2} \overline{A_j}\right)} =$$
$$\frac{\mathbb{P}(A_i)}{\mathbb{P}\left(\bigcap_{j\in S_1} \overline{A_j} \mid \bigcap_{j\in S_2} \overline{A_j}\right)} \leq \frac{x_i \prod_{ij\in E(D)} (1-x_j)}{\mathbb{P}\left(\bigcap_{j\in S_1} \overline{A_j} \mid \bigcap_{j\in S_2} \overline{A_j}\right)}.$$

The second equality follows from $A_i$ being mutually independent from $S_2$. Denote $S_1 = \{j_1, \ldots, j_r\}$. If $r = 0$, the denominator is 1 and the statement holds. Otherwise

$$\mathbb{P}\left(\overline{A_{j_1}} \cap \ldots \cap \overline{A_{j_r}} \mid \bigcap_{e\in S_2} \overline{A_e}\right) =$$
$$\left(1 - \mathbb{P}\left(A_{j_1} \mid \bigcap_{e\in S_2} \overline{A_e}\right)\right) \ldots \left(1 - \mathbb{P}\left(A_{j_r} \mid \overline{A_{j_1}} \cap \ldots \cap \overline{A_{r-1}} \cap \bigcap_{e\in S_2} \overline{A_e}\right)\right) \geq$$
$$(1-x_{j_1}) \ldots (1-x_{j_r}) = \prod_{ij\in E(D)\cap S} (1-x_j) \geq \prod_{ij\in E(D)} (1-x_j).$$

Thus the inductive statement holds. We conclude by computing

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) = (1 - \mathbb{P}(A_1)) \ldots \left(1 - \mathbb{P}\left(A_n \mid \overline{A_1} \cap \ldots \cap \overline{A_{n-1}}\right)\right) \geq \prod_{i=1}^{n} (1-x_i).$$

□

**Remark.** The symmetric version follows from the asymmetric version. Indeed, if $d = 0$, then the statement holds. If $d > 0$, then choosing $x_i = \frac{1}{d+1} < 1$ we have

$$x_i \prod_{j\in N(i)} (1-x_j) \geq \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^d > \frac{1}{(d+1)e} \geq p \geq \mathbb{P}(A_i).$$

## 6. Forbidding $C_4$

2026-11-10

**Theorem 27** (Erdős, 1938).

$$\mathrm{ex}(n, C_4) \leq \frac{n}{4}\left(\sqrt{4n-3}+1\right) \sim \frac{1}{2}n^{\frac{3}{2}}.$$

**Proof.** Consider a graph $G$ on $n$ vertices with no $C_4$. No two distinct copies of $P_3$ in $G$ have both ends in common, as otherwise they create a $C_4$. There are $\binom{n}{2}$ choices of the ends, and thus at most $\binom{n}{2}$ copies of $P_3$. The number of $P_3$'s with $v$ as the middle vertex is $\binom{\mathrm{d}(v)}{2}$. $\binom{x}{2}$ is convex, thus

$$\binom{n}{2} \geq \sum_{v\in V(G)} \binom{\mathrm{d}(v)}{2} \geq n\binom{\frac{1}{n}\sum \mathrm{d}(v)}{2} = n\binom{2e(G)/n}{2} = \frac{2e(G)^2}{n} - e(G).$$

Solving for $e(G)$ we get the statement. □

**Theorem 28** (Erdős, Rényi, Sós, 1966).

$$\mathrm{ex}(n, C_4) \sim \frac{1}{2}n^{\frac{3}{2}}.$$

**Proof.** For a prime number $p$ define a graph $G$ on the vertex set $\mathbb{Z}_p \times \mathbb{Z}_p \setminus \{(0,0)\}$, where distinct pairs $(x, y)$ and $(a, b)$ are connected exactly when $ax + by \equiv 1 \pmod p$. For fixed $(a, b)$ there are

exactly $p$ solutions $(x, y)$ to $ax + by \equiv 1 \pmod p$. Indeed, we can assume $b \neq 0$ (as one of $a, b$ is non-zero) and for any $x$ taking $y \equiv (1 - ax)b^{-1} \pmod p$ is the only possibility. It can happen that one of the solutions is $(a, b)$, which we discard so as not to make a loop in $G$. The degree of $(a, b)$ is at least $p - 1$. Thus $e(G) \geq \frac{1}{2}(p-1)\left(p^2-1\right) \sim \frac{1}{2}p^3 \sim \frac{1}{2}n^{\frac{3}{2}}$, where $n = p^2 - 1$ is the number of vertices. There are no $C_4$'s in $G$, because if vertices $(a, b)$ and $(a', b')$ have two common neighbours, then there are two different solutions to the system $ax+by \equiv 1 \pmod p$ and $a'x+b'y \equiv 1 \pmod p$, which is impossible.

This construction only works if $\sqrt{n+1}$ is prime. By the prime number theorem, there is a prime between $(1 - o(1))\sqrt{n}$ and $\sqrt{n}$, so we can construct an $n$-vertex graph by adding $n-\left(p^2-1\right)$ isolated vertices to the construction above. □

**Remark.** The following problem is what motivated Erdős to study $\mathrm{ex}(n, C_4)$.

**Theorem 29** (Erdős, 1938)**.** Let $A \subseteq \{1, \ldots, n\}$ be a set of integers such that all products of pairs of elements of $A$ are distinct. Then $|A| \leq \pi(n) + O\left(n^{\frac{3}{4}}\right)$, where $\pi(n)$ is the number of primes in $\{1, \ldots, n\}$.

**Proof** (with error term $O\left(n^{\frac{5}{6}}\right)$)**.** Let us start with two claims.

***Claim* 29.1.** If $G$ is a bipartite graph not containing $C_4$ with parts of sizes $a$ nad $b$, then $e(G) \leq a\sqrt{b} + b$.

*Proof.* We repeat the proof of Theorem 27, only now we count the $P_3$'s with endpoints in the part of size $a$. ◁

***Claim* 29.2.** Let $D = \left\{1, \ldots, \left\lfloor n^{\frac{2}{3}} \right\rfloor\right\}$ and let $B$ be the set containing $D$ and all primes between $n^{\frac{2}{3}}$ and $n$. Then any integer in $\{1, \ldots, n\}$ can be written as a product of an element of $D$ and an element of $B$.

*Proof.* Choose $x \in \{1, \ldots, n\}$. If $x \leq n^{\frac{2}{3}}$, then $x \in 1 \cdot D$. If $x$ has a prime factor $p$ larger than $n^{\frac{1}{3}}$, then $x = \frac{x}{p} \cdot p \in D \cdot B$. Now assume $x > n^{\frac{2}{3}}$ and every prime divisor of $x$ is at most $n^{\frac{1}{3}}$. Then $x$ has a divisor $q$ such that $n^{\frac{1}{3}} \leq q \leq n^{\frac{2}{3}}$. Then $x = q \cdot \frac{x}{q} \in D \cdot B$. ◁

For each $a \in A$ fix one representation $a = b \cdot d$ with $b \in B$, $d \in D$. Define a bipartite graph with parts $B$ and $D$ with an edge $bd$ when $bd$ is a representation of an element in $A$. There are $|A|$ edges. If we had a cycle $d_1b_1d_2b_2$, then we have $(b_1d_1)(b_2d_2) = (d_1b_2)(b_1d_2)$ and there exist two pairs of elements of $A$ with the same products.

We partition $B$ into parts: $B_1$ with elements less than $n^{\frac{1}{2}}$, $B_2$ with elements between $n^{\frac{1}{2}}$ and $n^{\frac{2}{3}}$ and $B_3$ with primes bigger than $n^{\frac{2}{3}}$. Similarly, we partition $D$ into parts: $D_1$ with elements less than $n^{\frac{1}{3}}$, $D_2$ with elements between $n^{\frac{1}{3}}$ and $n^{\frac{1}{2}}$ and $D_3$ with elements bigger than $n^{\frac{1}{2}}$. Vertices in $B_2$ are not adjacent to vertices in $D_3$ and vertices in $B_3$ are not adjacent to vertices in $D_2 \cup D_3$. We have

$$e(D_1, B) \leq n^{\frac{1}{3}}\left(\pi(n) + n^{\frac{2}{3}}\right)^{\frac{1}{2}} + \pi(n) + n^{\frac{2}{3}} \leq O\left(n^{\frac{5}{6}}\right) + \pi(n) + n^{\frac{2}{3}},$$

$$e(B_1, D) \leq n^{\frac{1}{2}}\left(n^{\frac{2}{3}}\right)^{\frac{1}{2}} + n^{\frac{2}{3}} = n^{\frac{5}{6}} + n^{\frac{2}{3}},$$

$$e(D_2, B_2) \leq n^{\frac{1}{2}}\left(n^{\frac{2}{3}}\right)^{\frac{1}{2}} + n^{\frac{2}{3}} = n^{\frac{5}{6}} + n^{\frac{2}{3}}.$$

This implies $|A| = e(G) \leq \pi(n) + O\left(n^{\frac{5}{6}}\right)$. □

**Remark.** A proof with error term $O\left(n^{\frac{3}{4}}\right)$ requires more careful splitting of the edges.

## 7. Forbidding a biclique

2026-11-17

**Theorem 30** (Kővári, Sós, Turán, 1954). For any integers $s, t \geq 1$ we have

$$\mathrm{ex}(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n.$$

**Proof.** We count $s$-stars, that is, pairs $(v, S)$ where $v$ is a vertex and $S$ is a set of neighbours of $v$ of size $s$. Any set of $s$ vertices has at most $t-1$ common neighbours, as otherwise we have a $K_{s,t}$, so there are at most $(t-1)\binom{n}{s} \leq (t-1)\frac{n^s}{s!}$ such pairs. On the other hand, for every vertex $v$ there are exactly $\binom{\mathrm{d}(v)}{s}$ stars centred around it and

$$\sum_{v \in V(G)} \binom{\mathrm{d}(v)}{s} \geq n \binom{\frac{1}{n}\sum \mathrm{d}(v)}{s} = n\binom{2e(G)/n}{s} \geq n\frac{(2e(G)/n - s + 1)^s}{s!}.$$

This gives the desired bound. □

**Theorem 31** (Erdős, Newman, 1977). Given $\varepsilon > 0$ and $k$ large enough, if $A$ is a set of non-negative integers such that $\{1^2, \ldots, k^2\} \subseteq A + A$, then $|A| \geq k^{\frac{2}{3}-\varepsilon}$.

**Proof.** Let $G$ be a graph on vertex set $A$ with an edge $aa'$ when $a + a'$ is a square of a number from $1, \ldots, k$. $G$ has at least $\frac{k}{2}$ edges, as every odd number from $1^2, \ldots, k^2$ is a sum of two different elements of $A$. We will show that $G$ has no $K_{2,t}$ for appropriately chosen $t$. Fix $a, a' \in A$ with $a > a'$ and let $b$ be a common neighbour of $a, a'$. This means that $a + b = x^2$ and $a' + b = y^2$, so $a - a' = x^2 - y^2 = (x-y)(x+y)$. Notice that $x - y$ is a divisor of $a - a'$ and it uniquely determines $x+y$, and thus also $x, y$ and $b$. This means that $a$ and $a'$ have at most $\sigma(a - a')$ common neighbour, where $\sigma(m)$ denotes the number of divisors of $m$. We know that $\sigma(m) = o\big(m^{\delta}\big)$for any $\delta > 0$. We have $a - a' \leq k^2$ (we can assume $A$ has no element greater than $k^2$), so there exists a constant $C$ such that $\sigma(a - a') \leq Ck^{2\varepsilon}$ for any pair $a, a'$. For $k$ large enough, the number of common neighbours of a pair of vertices is less that $k^{3\varepsilon}$, so for $t = \lfloor k^{3\varepsilon} \rfloor$ the graph $G$ is $K_{2,t}$-free. By Theorem 30 we have

$$\frac{k}{2} \leq e(G) \leq \frac{1}{2}(t-1)^{\frac{1}{2}}|A|^{\frac{3}{2}} + \frac{1}{2}|A| \leq \frac{1}{2}k^{\frac{3\varepsilon}{2}}|A|^{\frac{3}{2}}.$$

□

**Theorem 32.** For any integers $s, t \geq 2$ and $n$ large enough we have

$$\mathrm{ex}(n, K_{s,t}) \geq \frac{1}{4}(s!t!)^{\frac{1}{st}} n^{2-\frac{1}{s}-\frac{1}{t}}.$$

**Proof.** Fix $n, e$ with $st \leq e \leq \binom{n}{2}$. There are $\binom{\binom{n}{2}}{e}$ graphs on $n$ vertices and $e$ edges. Now we count the number of $K_{s,t}$'s we can find in such graphs. There are $\binom{n}{s+t}\binom{s+t}{s} \leq \frac{n^{s+t}}{s!t!}$ ways to fix a $K_{s,t}$ in an $n$-vertex graph. The remaining edges can be chose in $\binom{\binom{n}{2}-st}{e-st}$ ways. If we have $\binom{\binom{n}{2}}{e} > \frac{n^{s+t}}{s!t!}\binom{\binom{n}{2}-st}{e-st}$, then there exists an $n$-vertex graph without $K_{s,t}$ and with $e$ edges. By expanding we see this is equivalent to

$$\frac{\binom{n}{2}\cdots\left(\binom{n}{2}-st+1\right)}{e\cdots(e-st+1)} > \frac{n^{s+t}}{s!t!}.$$

We have

$$\frac{\binom{n}{2}\cdots\left(\binom{n}{2}-st+1\right)}{e\cdots(e-st+1)} > \left(\frac{\binom{n}{2}}{e}\right)^{st} > \left(\frac{n^2}{4e}\right)^{st}.$$

Thus, it is enough to have $\frac{n^2}{4e} \geq \frac{n^{\frac{1}{s}+\frac{1}{t}}}{(s!t!)^{\frac{1}{st}}}$. □