

# HYPERGRAPH THEORY AND APPLICATIONS

NOTES

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*„Modulo the fact I can be completely wrong, it basically works like that.”*

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## 1. Examples of hypergraphs

**Definition 1.** A (simple) hypergraph is a pair  $(V, E)$ , where  $V$  is a set and  $E \subseteq \mathcal{P}(V)$ . We call the elements of  $V$  vertices and the elements of  $E$  edges. In particular, we allow  $\emptyset$  and singletons to be edges. We assume  $V, E \neq \emptyset$ .

Multihypergraphs are hypergraphs with multiedges –  $E$  is a multiset, not a set.

**Definition 2.** Let  $H$  be a hypergraph and  $v \in V(H)$ . We define the degree of  $v$  as  $d_H(v) = |\{e \in E : v \in e\}|$ .  $\Delta(H)$  is the maximal degree of a vertex and  $\delta(H)$  is the minimal degree.

A hypergraph is  $t$ -regular if all its vertices are of degree  $t$ .

**Definition 3.** A hypergraph is  $r$ -uniform if all edges are sets of size  $r$ . In particular 2-uniform hypergraphs are graphs. We call  $r$ -uniform hypergraphs  $r$ -graphs.

**Definition 4.** Two hypergraphs are isomorphic if there exists an edge preserving bijection between the vertices of the hypergraphs.

**Example.** The Fano plane is a hypergraph on 7 vertices with edges like in the figure.

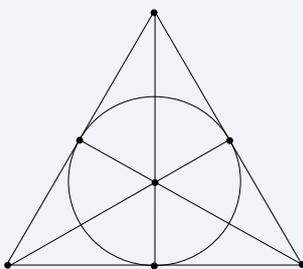


Figure 1: The Fano plane.

**Definition 5.** The incidence matrix of  $H = (V, E)$  with  $|V| = n$  and  $|E| = m$  is an  $m \times n$  matrix given by  $a_{ij} = \begin{cases} 1 & v_j \in e_i \\ 0 & v_j \notin e_i \end{cases}$ .

**Proposition 1.** For a hypergraph  $H = (V, E)$  we have  $\sum_{v \in V} d(v) = \sum_{e \in E} |e|$ .

**Proof.** We count the number of 1's in the incidence matrix by rows and by columns.  $\square$

**Definition 6.** For a hypergraph  $H$  with an incidence matrix  $M$  we define its dual hypergraph  $H^*$  as the hypergraph with the incidence matrix  $M^T$ .

**Proposition 2.** 1.  $(H^*)^* = H$ .

2. There exists a bijection  $f : V(H) \rightarrow E(H^*)$ .
3. There exists a bijection  $g : E(H) \rightarrow V(H^*)$ .
4. For  $x \in e$  in  $H$  we have  $g(e) \in f(x)$  in  $H^*$ .
5.  $f(x) = |f(x)|$  and  $d(g(e)) = |e|$ .
6. The dual of a  $t$ -regular hypergraph is  $t$ -uniform.
7. The dual of a  $t$ -uniform hypergraph is  $t$ -regular.

**Definition 7.** The intersection graph (line graph) of a hypergraph  $H = (V, E)$  is a graph on  $E$  with an edge between  $e, f \in E$  exactly when  $e \cap f \neq \emptyset$ .

**Proposition 3.** Every simple graph is an intersection graph of some hypergraph.

**Proof.** For a graph  $G = (V, E)$  we define a hypergraph  $H$  with  $V(H) = E$  and the edge set  $E(H) = \{\{e : v \in e \in E\} : v \in V\}$ .  $\square$

**Definition 8.** Let  $k \geq 1$  and let  $x_1, \dots, x_k$  be distinct vertices of a hypergraph and  $e_1, \dots, e_{k-1}$  be distinct edges. We say  $P_k = x_1 e_1 \dots x_{k-1} e_{k-1} x_k$  is a (Berge) path if  $x_i, x_{i+1} \in e_i$  for every  $i \in [k-1]$ . The length of the path is  $k-1$ . In this course, we call  $x_1, \dots, x_k$  the defining vertices and  $e_1, \dots, e_{k-1}$  the defining edges.

A cycle is a path (with  $k \geq 2$ ) such that there exists  $e_k \in E \setminus \{e_1, \dots, e_{k-1}\}$  with  $x_k, x_1 \in e_k$ . The length of the cycle is  $k$ .

**Definition 9.** The girth of  $H$ , denoted as  $g(H)$ , is the minimal length of a cycle in  $H$ . We write  $g(H) = \infty$  if  $H$  does not contain a cycle.

**Definition 10.** We write  $K_n^r$  for the  $n$ -vertex  $r$ -uniform complete hypergraph, so  $K_n^r = \left([n], \binom{[n]}{r}\right)$ .

**Definition 11.** We say  $H = (V, E)$  is a  $k$ -partite hypergraph if there exists a partition  $V = V_1 \sqcup \dots \sqcup V_k$  such that for each  $e \in E$  we have  $|e \cap V_i| \leq 1$ .

**Definition 12.** We say  $H = (V, E)$  is intersecting if for all  $e_1, e_2 \in E$  we have  $e_1 \cap e_2 \neq \emptyset$ .

**Definition 13.** Let  $t, k, n \in \mathbb{N}$  be such that  $2 \leq t < k < n$ . We say a hypergraph is a  $(t, k, n)$ -Steiner system (denoted  $(S(t, k, n))$ ) if it is an  $n$ -vertex  $k$ -graph such that for each subset of vertices of size  $t$  there exists a unique edge containing it.

**Definition 14.** A linear space is a hypergraph such that each pair of vertices is contained in precisely one edge. We assume there are no empty and singleton edges. A trivial linear space contains only one edge with all the vertices.

**Remark.** The Fano plane is an example of  $S(2, 3, 7)$  and a linear space.

**Definition 15.** A linear space is planar if it is possible to draw its vertices as points of the plane so that the edges are precisely the lines (maximal sets of collinear points).

**Remark.** The 2-vertex planar linear space is a line. The 3-vertex planar linear space is a line or a triangle. For 4 vertices we have a line, a near pencil (3 in a line and one outside of it), and a square. For 5 vertices we have a line, a near pencil, a pentagon, a wedge, and three vertices in one line and two in a second one.

**Theorem 1 (Sylvester, Gallai).** In every non-trivial planar linear space there is a line intersecting exactly two points.

**Proof.** Consider all lines and points. Choose a line and a point such that their distance is minimal (but non-zero). If the line contains three points, we can find a line and a point with smaller distance.  $\square$

**Corollary.** The Fano plane is not a planar linear space.

**Proof.** The Fano plane does not have an edge of size 2.  $\square$

**Theorem 2.** If a non-trivial linear space has  $n$  vertices and  $m$  edges, then  $m \geq n$ .

**Proof.** Assume  $H$  is a linear space and  $m \leq n$ . If  $v \in V$ ,  $e \in E$  and  $v \notin e$ , then  $d(v) \geq |e|$  as no two vertices in  $e$  are connected to  $v$  by the same edge. Thus  $nd(v) \geq m|e|$  and  $\frac{1}{n(m-d(v))} \geq \frac{1}{m(n-|e|)}$

(note the denominators are non-zero). This gives us

$$1 = \sum_{v \in V} \frac{m - d(v)}{n(m - d(v))} = \sum_{v \in V} \sum_{e \ni v} \frac{1}{n(m - d(v))} \geq \sum_{e \in E} \sum_{v \in e} \frac{1}{m(n - |e|)} = \sum_{e \in E} \frac{n - |e|}{m(n - |e|)} = 1.$$

Thus the inequalities we used are not strict and  $n = m$ . This bound is strict – a near pencil is an example.  $\square$

**Theorem 3.** Let  $H$  be an intersecting linear space. If  $H$  is non-trivial and is not the near pencil, then there exists  $k$  such that  $|E| = |V| = k^2 + k + 1$  and  $H$  is  $(k + 1)$ -uniform and  $(k + 1)$ -regular.

**Proof.** Assume two edges  $e, f$  cover all the vertices. Those edges intersect in one vertex. If both  $e$  and  $f$  contain two other vertices (say  $a, b$  and  $c, d$ ), then  $\{a, c\}$  and  $\{b, d\}$  are edges of the hypergraph that do not intersect. Thus one of  $e, f$  contains two vertices and the hypergraph is a pencil. This proved no two edges cover all vertices.

Let us take the edges  $e, f$  and a vertex  $p \in V \setminus (e \cup f)$ . Let  $x \in e$ . There is an edge connecting  $p$  and  $x$ . This edge contains a vertex of  $f$  – some  $g(x)$ . This defines a bijection between  $e$  and  $f$ . Thus  $|e| = |f| = k + 1$ . Applying the same reasoning to the dual hypergraph, we get that all the vertices are of degree  $\ell + 1$ . We can show  $k + 1 \geq d(p) \geq k + 1$ , so  $k = \ell$ .

By uniformity and regularity we have  $n(k + 1) = m(k + 1)$ , so  $n = m$  and by  $\binom{n}{2} = \binom{k+1}{2}m$  (every pair of vertices is covered by exactly one edge) the other equality follows.  $\square$

## 2. Chromatic number

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**Definition 16.** A colouring of the vertices of a hypergraph  $H$  is a colouring such that no edge with at least two vertices is monochromatic. The chromatic number  $\chi(H)$  is the minimal number of colours  $k$  such that a proper  $k$ -colouring exists.

**Proposition 4.**

$$\chi\left(K_{(n-1)(k-1)+1}^k\right) = n$$

**Proof.** We can colour at most  $k - 1$  vertices with one colour. By the pigeon-hole principle every colouring with  $n - 1$  colours has at least  $k$  vertices in one colour. If we colour with  $n$  colours, it is possible to divide the vertices into  $n$  groups of size at most  $k - 1$ .  $\square$

**Proposition 5 (Zykov construction).** Take  $G_1 = K_1$ . Assume  $G_1, \dots, G_i$  are defined. To construct  $G_{i+1}$  take disjoint copies of  $G_1, \dots, G_i$ . For each tuple  $(v_1, \dots, v_i) \in V(G_1) \times \dots \times V(G_i)$  create a new vertex  $v$  connected exactly to  $v_1, \dots, v_i$ . We have  $\chi(G_i) = i$  and each  $G_i$  is triangle-free.

**Proof.** When colouring  $G_{i+1}$  there needs to be a colour in the copy of  $G_{i+1}$  that does not appear in the copy of  $G_t$ . This allows us to create a tuple of  $i$  vertices with different colours. The vertex connected to them must be a different colour. Thus  $\chi(G_{i+1}) \geq i + 1$ . We can show that  $\chi(G_{i+1}) \leq i + 1$  by colouring the new vertices with one new colour. It is easy to check each  $G_i$  is triangle-free.  $\square$

**Proposition 6 (Mycielski construction).** Take  $M_1 = K_1, M_2 = K_2$ . To construct  $M_i$  assume  $M_{i-1}$  is defined. Take a copy of  $M_{i-1}$ . Copy each vertex of the  $M_{i-1}$  (without any edges) and connect the copy to the neighbours of the copied graph. Add a new vertex and connect it to the copied vertices. We have  $\chi(M_i) = i$  and each  $M_i$  is triangle-free.

**Proof.** We have  $\chi(M_{i+1}) \geq i + 1$  – a proper colouring with  $i$  colours would use  $i - 1$  colours on the copy of  $M_i$  and that would be a proper  $(i - 1)$ -colouring of  $M_i$ , a contradiction.  $\chi(M_{i+1}) \leq i + 1$  follows from an inductive colouring – colour  $M_i$ , colour the copy the same and give a new colour to the new vertex. It is easy to check each  $M_i$  is triangle-free.  $\square$

**Proposition 7 (Tutte construction).** Take  $T_1 = K_1$ . To construct  $T_{i+1}$  assume  $T_i$  is defined. Let

$n_i = |V(T_i)|$ . Take an independent set  $I$  of size  $(n_i - 1)i + 1$ . For each subset  $X \subseteq I$  of size  $n_i$  take a copy of  $T_i$  and connect its vertices with the vertices of  $X$  to form a perfect matching. We have  $\chi(T_i) = i$  and each  $T_i$  is triangle-free.

**Proof.** We can show  $\chi(T_{i+1}) \leq i + 1$  inductively by colouring  $I$  with one colour. If we have a colouring of  $T_{i+1}$  with  $i$  colours by the pigeon-hole principle there exists a monochromatic  $n_i$ -subset of  $I$ . The copy of  $T_i$  connected to that set must be coloured with  $i - 1$  colours – contradiction. It is easy to check each  $T_i$  is triangle-free.  $\square$

**Proposition 8** (Shift graphs; Erdős, Hajnal). Take  $V(S_n) = \{(i, j) : 1 \leq i < j \leq n\}$  and  $E(S_n) = \{(p, q)(r, s) : q = r \vee p = s\}$ . We have  $\chi(S_n) = \lceil \log_2 n \rceil$  and each  $S_n$  is triangle-free.

**Proof.** We have  $\chi(S_n) \leq \lceil \log_2 n \rceil$  by colouring the first half  $(i, j \leq \frac{n}{2})$  with  $\lceil \log_2 \frac{n}{2} \rceil$  colours, the other half with the same colours, and the intervals crossing the middle point with one colour. Let  $\varphi$  be a proper  $k$ -colouring. Define  $X_i = \{\varphi((i, j)) : j > i\}$ . We have  $X_i \neq X_j$  for  $i < j$  as  $X_i$  contains  $\varphi((i, j))$  and no interval  $(j, k)$  can have that colour. Thus we have  $n$  sets, so  $n \leq 2^k$ . It is easy to check each  $S_n$  is triangle-free.  $\square$

**Proposition 9** (Kneser graphs). Let us fix  $n, k$ . We define  $KN(n, k)$  to be the graph on the vertex set  $\binom{[2n+k]}{n}$  with an edge between two sets if they are disjoint. We have  $\chi(KN(n, k)) \leq k + 2$ .

**Proof.** For each  $i \in [k + 1]$  we colour the sets containing  $i$  with one colour. We are left with  $n$ -subsets of a  $2n - 1$  element set. They all intersect, so we can colour them with one colour.  $\square$

**Theorem 4** (Lusternik, Schnirelmann). If the  $k$ -dimensional sphere  $\mathbb{S}^k = \{x \in \mathbb{R}^{k+1} : \|x\| = 1\}$  is covered with  $k + 1$  clopen (closed or open) sets, then one of these sets contains a pair of antipodal points  $x$  and  $-x$ .

**Theorem 5** (Kneser conjecture).

$$\chi(KN(n, k)) \geq k + 2.$$

**Proof** (Lovász, 1978). Place  $2n + k$  points on  $\mathbb{S}^{k+1}$  in general position (such that no  $k + 2$  point lie on any equator of  $\mathbb{S}^{k+1}$ ). For  $x \in \mathbb{S}^{k+1}$  denote by  $H(x)$  the open hemisphere with  $x$  as its pole. Suppose  $A_1, \dots, A_{k+1}$  is a partition of the  $n$ -element subsets of the  $2n + k$  points such that the sets in every  $A_i$  pairwise intersect. Let  $U_i = \{x \in \mathbb{S}^{k+1} : \exists X \in A_i, X \subseteq H(x)\}$ . Those sets are open and together with  $C = \mathbb{S}^{k+1} \setminus (U_1 \cup \dots \cup U_{k+1})$  cover  $\mathbb{S}^{k+1}$ .

One of those sets contains two antipodal points  $x, -x$ . It cannot be  $C$ , because that would mean  $H(x)$  and  $H(-x)$  both contain at most  $n - 1$  of our chosen points, so at least  $k + 2$  points lie on the equator  $\overline{H(x) \cap H(-x)}$ . This means we have  $x, -x \in U_i$  for some  $i$ . This means there are sets  $X, X' \in A_i$  such that  $X \subseteq H(x)$  and  $X' \subseteq H(-x)$ . As  $H(x) \cap H(-x) = \emptyset$ , we arrive at a contradiction.  $\square$

**Corollary.**  $KN(n, n - 1)$  is a triangle-free graph with  $\chi(KN(n, n - 1)) = n + 1$ .

**Theorem 6.** By a  $[p, k, n]$ -hypergraph we mean a  $k$ -graph with girth at least  $p$  and chromatic number at least  $n$ . A  $[p, k, n]$ -hypergraph exists for any  $p, k, n$ .

**Proof.** We will prove that there exists a  $[p, k, n]$ -hypergraph that is  $s$ -partite for  $s = (k - 1)(n - 1) + 1$ . The base case  $p = 1$  is  $K_s^k$ .

Let us fix  $p > 1, k, n$ . Let  $s = (k - 1)(n - 1) + 1$ . We will inductively define hypergraphs  $H_1, \dots, H_{s+1}$  that are  $s$ -partite.

To construct  $H_1$  we take disjoint sets  $V_1, \dots, V_s$  of vertices of size  $\binom{s-1}{k-1}$ . We create hyperedges in such a way that for any choice of  $k$  sets from  $V_1, \dots, V_s$  there exists a hyperedge with exactly one vertex in each of the chosen sets. We can make those edges disjoint (the size of  $V_i$  makes that possible).

Assume  $H_j$  is already constructed. Let  $X_1, \dots, X_s$  be the  $s$ -partition of  $H_j$ . Let  $K_j = |X_j|$ . Let  $F_j$  be a  $[p - 1, K_j, n]$ -hypergraph with vertices  $V_j$  and edges  $E_j$ . Take  $|E_j|$  vertex disjoint copies of

$H_j$ . Now we will define  $H_{j+1}$  together with its partition  $X'_1, \dots, X'_s$ . For  $i \neq j$  let  $X'_i$  be the sum of the  $X_i$ 's from the copies of  $H_j$ . We set  $X'_j = V_j$ . As  $K_j = |X_j|$ , we can treat the vertices in one edge  $E \in E_j$  as the  $X_j$  of the  $H_j$  corresponding to  $E$ . That is, all the edges of this copy of  $H_j$  that contain a vertex of  $X_j$  now contain a corresponding vertex of  $E$ . This is the construction of  $H_{j+1}$ .

Assume  $H_{s+1}$  is  $(n-1)$ -colourable and  $\varphi$  is a colouring witnessing that. Let  $X_1, \dots, X_s$  be the  $s$ -partition of  $H_{s+1}$ . As  $F_s$  is not  $(n-1)$ -colourable, there exists an edge of  $F_s$  that is monochromatic under  $\varphi$  (notice that it is not an edge of  $H_{s+1}$ ). This means that in  $H_{s+1}$  there exists a copy of  $H_s$  such that its  $s$ -th partition set is monochromatic. Repeating this argument for this  $H_s$  and its  $(s-1)$ -th partition set we obtain a copy of  $H_{s-1}$  such that its last two partition sets are both monochromatic. At the end we obtain  $H_1$  such that all of its  $s = (k-1)(n-1) + 1$  partition sets are monochromatic. This means that there are  $k$  partition sets of the same colour. This is a contradiction, as there exists an edge with one vertex in each of these sets.

Now we will show inductively that  $H_j$  has no cycle of length less than  $p$ . There are no cycles in  $H_1$ . Assume we have a cycle  $C$  in  $H_{j+1}$ . If it's contained in one copy of  $H_j$ , then it has length at least  $p$ . Assume it's not. This means  $C$  contains a vertex  $v$  from the  $j$ -th partition  $V_j$  set of  $H_{j+1}$ , which is the vertex set of  $F_j$ . The only edges  $v$  is in connect it to different copies of  $H_j$ . We go from  $v$  to some copy of  $H_j$ , move in this copy and come back to  $V_j$ . Now we move to a different copy of  $H_j$  – this corresponds to moving along an edge in  $F_j$ . As  $F_j$  has girth at least  $p$ , we must move between different copies of  $H_j$  at least  $p$  times before we come back to the starting vertex.  $\square$