Probabilistic Algorithms

Notes

„Modulo the fact I can be completely wrong, it basically works like that.”

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Contents

1. First moment method

2026-02-26

Idea (Probabilistic method). To show that an object $u \in E$ with property P exists, we build a probability distribution on E such that $\mathbb{P}(u \text{ satisfies } P) > 0$.

Idea (First moment method). We have the Markov inequality $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$. The first moment method uses this inequality to give results about existence of structures. Most importantly, if $\mathbb{E}[X] \leq t$, then $\mathbb{P}(X \leq t) > 0$.

Theorem 1 (Erdős, 1947). If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. It follows that $R(k,k) > \left\lfloor 2^{\frac{k}{2}} \right\rfloor$ for $k \geq 3$.

Proof. Consider a random uniform 2-colouring of K_n. For every $S \subseteq V(K_n)$ of size k we define $X_S = \begin{cases} 1 & S \text{ is monochromatic} \\ 0 & \text{otherwise} \end{cases}$, $X = \sum_{S \subseteq V(K_n), |S|=k} X_S$.

$\mathbb{P}(S \text{ is monochromatic}) = \mathbb{E}[X_S] = 2 \cdot 2^{-\binom{k}{2}}$. Thus $\mathbb{E}[X] = \binom{n}{k}2^{1-\binom{k}{2}} < 1$.

By first moment method, $\mathbb{P}(X = 0) > 0$. It follows that $R(k,k) > n$.

For $k \geq 3$ we take $n = \left\lfloor 2^{\frac{k}{2}} \right\rfloor$ and compute $\binom{n}{k}2^{1-\binom{k}{2}} \leq \frac{n^k}{k!}2^{1-\binom{k}{2}} \leq \frac{n^k}{k!}\frac{2^{k/2+1}}{2^{k^2/2}} < 1$. □

Definition 1 (Erdős-Rényi model). We write $G_{n,m}$ to denote the uniform distribution on all graphs with n vertices and m edges.

Theorem 2 (Erdős, 1959). For all k there exists a triangle-free graph G with $\chi(G) > k$.

Proof. We take $G \sim G_n^p$ (n vertices, each edge exists with probability p independently) with $p = n^{-\frac{2}{3}}$.

Given $S \subseteq V$ with $|S| = \left\lceil \frac{n}{2k} \right\rceil$ let us define I_S as the indicator of whether S is an independent set. Let $I = \sum_{|S|=\left\lceil \frac{n}{2k} \right\rceil} I_S$.

$$\mathbb{E}[I] = \sum_S \mathbb{E}[I_S] = \binom{n}{\left\lceil \frac{n}{2k} \right\rceil}(1-p)^{\binom{\left\lceil \frac{n}{2k} \right\rceil}{2}} \leq 2^n (1-p)^{\binom{\left\lceil \frac{n}{2k} \right\rceil}{2}} \leq 2^n e^{-p\binom{\left\lceil \frac{n}{2k} \right\rceil}{2}} \leq 2^n e^{-n^{2/3}\frac{n^2}{16k^2}} \to 0$$

Thus for large enough n we have $\mathbb{P}(I > 0) < \frac{1}{2}$ and $\mathbb{P}\left(\alpha(G) < \frac{n}{2k}\right) > 0$.

Given vertices x, y, z we have $\mathbb{P}(xyz \text{ is a triangle}) = p^3 = n^{-2}$. Let T be the number of triangles in G. $\mathbb{E}[T] = \binom{n}{3}n^{-2} \leq \frac{n^3}{3!}n^{-2} = \frac{n}{6}$. By Markov's inequality $\mathbb{P}\left(T \geq \frac{n}{2}\right) \leq \frac{1}{3}$.

$\mathbb{P}\left(T \leq \frac{n}{2} \wedge I = 0\right) > 0$, because those events must intersect as their probability is high. We choose $\frac{n}{2}$ vertices that touch all triangles and delete them. Let G' be the obtained triangle-free graph.

As monochromatic sets are independent, we have $\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n}{2\alpha(G)} \geq k$. □

Theorem 3. For every n, $R(k,k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$. Hence $R(k,k) > \frac{1}{e}k2^{\frac{k}{2}}(1 + o(1))$.

Proof. Consider a random uniform 2-edge colouring of K_n. Let I count the number of monochromatic cliques of size k. We have $\mathbb{E}[I] = \binom{n}{k}2^{1-\binom{k}{2}} =: m$. By first moment method, $\mathbb{P}(I \leq m) > 0$. So there exists a colouring γ such that there are at most m monochromatic cliques.

We remove one vertex from each clique, we end up with a colouring γ' of K_{n-m} with no monochromatic cliques. Thus, the first part of the theorem statement holds. To prove the second one, we need to maximise $n - \binom{n}{k}2^{1-\binom{k}{2}}$. □

Theorem 4. Let G be an n-vertex graph with at most $\frac{nd}{2}$ edges for some fixed $d \geq 1$. Then $\alpha(G) \geq \frac{n}{2d}$.

Proof. Let S be a subset of vertices, each vertex taken into S independently with probability p. Let $X = |S|$ and let Y be the number of edges induced by S. We have $\mathbb{E}[Y] \leq \frac{nd}{2}p^2$. Thus $\mathbb{E}[X - Y] \geq np - \frac{nd}{2}p^2$. Let us take $p = \frac{1}{d}$. We get $\mathbb{E}[X - Y] \geq \frac{n}{2d}$. By first moment method, there exists S with $|S| - |E(G[S])| \geq \frac{n}{2d}$.

By removing one endpoint from every edge of S we get an independent set S' with $|S'| \geq \frac{n}{2d}$. □

Definition 2. A dominating set $S \subseteq V(G)$ is a set such such that $S \cup N(S) = V(G)$.

Theorem 5. Let G be an n-vertex graph with minimum degree $\delta > 1$. Then G has a dominating set of size at most $n\frac{1+\ln(\delta+1)}{\delta+1}$.

Proof. Let S be a random set of vertices of G (taken independently with probability p). Let $X = |V(G) \setminus N[S]|$. We have $\mathbb{P}(v \in X) = (1-p)^{\deg(v)+1} \leq (1-p)^{\delta+1}$. Let $D = S \cup X$. We have

$$\mathbb{E}[D] \leq \mathbb{E}[|S|] + \mathbb{E}[|X|] \leq np + n(1-p)^{\delta+1} \leq n\left(p + e^{-p(\delta+1)}\right).$$

Maximizing the function on the right we get $p = \frac{\ln(\delta+1)}{\delta+1}$. We conclude by first moment method, as D is dominating. □

2. Second moment method

2026-03-09

Proposition 1 (Chebyshev's inequality)**.** For every $t > 0$ and non-constant X we have

$$\mathbb{P}(|X - \mathbb{E}[X]| > t) \leq \frac{\mathrm{Var}(X)}{t^2}.$$

Theorem 6 (Hardy, Ramanujan 1920)**.** Let $\omega(n) \to +\infty$. Let $v(x)$ be the number of prime factors of x. The number of $x \in [n]$ such that $|v(x) - \ln\ln n| > \omega(n)\sqrt{\ln\ln n}$ is $o(n)$.

Proof (Turán 1934)**.** Let x be chosen uniformly from $[n]$. For prime p let us set $X_p = \begin{cases} 1 & p \mid x \\ 0 & p \nmid x \end{cases}$. $X = \sum_{p\in\mathbb{P}} X_p$ counts the prime factors of x. Let us set $Y = \sum_{p\in\mathbb{P}: p<n^{\frac{1}{10}}} X_p$. There are no more than 10 divisors of x that are greater than $n^{\frac{1}{10}}$. Thus $v(x) - 10 \leq Y \leq v(x)$.

We have $\mathbb{E}[X_p] = \frac{\lfloor \frac{n}{p} \rfloor}{n} = \frac{1}{p} + O\left(\frac{1}{n}\right)$. Thus $\mathbb{E}[Y] = \sum_{p\in\mathbb{P}:p<n^{\frac{1}{10}}} \left(\frac{1}{p} + O\left(\frac{1}{n}\right)\right) = \ln\ln n + O(1)$. We have $\mathrm{Var}(X_p) = \mathbb{E}[X_p] - \mathbb{E}\left[X_p^2\right] = \frac{1}{p} + O\left(\frac{1}{n}\right) - \frac{1}{p^2} + O\left(\frac{1}{n}\right) = \frac{1}{p} - \frac{1}{p^2} + O\left(\frac{1}{n}\right)$ and $\mathrm{Cov}(X_p, X_q) = \mathbb{E}[X_pX_q] - \mathbb{E}[X_p]\mathbb{E}[X_q] = \frac{\lfloor \frac{n}{pq} \rfloor}{n} - \frac{\lfloor \frac{n}{p} \rfloor}{n}\frac{\lfloor \frac{n}{q} \rfloor}{n} \leq \frac{1}{n}\left(\frac{1}{p} + \frac{1}{q}\right)$. Thus

$$\mathrm{Var}(Y) = \sum_p \mathrm{Var}(X_p) + \sum_{p\neq q} \mathrm{Cov}(X_p, X_q) \leq \sum_p \left(\frac{1}{p} - \frac{1}{p^2} + O\left(\frac{1}{n}\right)\right) + \sum_{p\neq q} \frac{1}{n}\left(\frac{1}{p} + \frac{1}{q}\right) \leq$$
$$\ln\ln n + O(1) + O(1) + O\left(\frac{\ln\ln n}{n}\right) \leq \ln\ln n + O(1).$$

By Chebyshev's inequality $\mathbb{P}\left(|Y - \ln\ln n| > \omega(n)\sqrt{\ln\ln n}\right) \leq \frac{\mathrm{Var}(Y)}{\omega(n)^2 \ln\ln n} \leq \frac{1}{\omega(n)^2} + o(1) = o(1)$. □

Theorem 7 (Erdős, Kac 1940)**.** For all $\lambda \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{x \in [n] : v(x) \geq \ln\ln n + \lambda\sqrt{\ln\ln n}\right\}\right| = \int_\lambda^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, \mathrm{d}t.$$

this means $v(x)$ behaves like a normal distribution.

Proposition 2 (Chernoff's bound). Let $X \sim \mathrm{Bin}(n, p)$. Then for any $0 \le t \le np$ we have

$$\mathbb{P}(|X - \mathbb{E}[X]| > t) \le 2e^{-\frac{t^2}{3np}}.$$

Remark. Hadwiger conjecture from 1943 states that if $\chi(G) \ge t$ then G contains a K_t minor

Hajós conjecture from 1981 states that if $\chi(G) \ge t$ then G contains a K_t subdivision.

Theorem 8 (Caitlin 1979). Hajós conjecture does not hold.

Proof (Erdős, Fajtlowicz 1981). Take $G \sim G\left(n, \frac{1}{2}\right)$. To show that $\chi(G) \ge \frac{n}{2\log_2 n}$ with high probability we will show that with high probability $\alpha(G) \le 2\log_2 n$. Let X be the number of independent sets of size $2\log_2 n := k$. $\mathbb{E}[X] = \binom{n}{k}2^{-\binom{k}{2}} \le \frac{n^k}{k!}2^{-\frac{k(k-1)}{2}} \le \frac{n^k}{k!}\left(2^{-\frac{k}{2}}\right)^{k-1} \le \frac{n}{k!} < \frac{1}{n}$. By Markov's inequality $\mathbb{P}\left(\chi(G) < \frac{n}{2\log_2 n}\right) < \frac{1}{n}$.

Now we will show that with high probability G has no K_ℓ subdivision with $\ell = \lceil 8\sqrt{n}\rceil$. We say that a set of vertices U is a $\frac{3}{4}$-clique if $e(G[U]) \ge \frac{3}{4}\binom{|U|}{2}$.

Let U be a set of ℓ vertices. $Y = e(G[U])$. We have $Y \sim \mathrm{Bin}\left(\binom{\ell}{2}, \frac{1}{2}\right)$. Thus $\mathbb{P}\left(Y \le \frac{3}{4}\binom{\ell}{2}\right) \le \mathbb{P}\left(\left|Y - \frac{\binom{\ell}{2}}{2}\right| \ge \frac{1}{4}\binom{\ell}{2}\right) \le 2e^{-\frac{5}{4}n}$, where the last inequality is the Chernoff's bound combined with some simplifications. We have

$$\mathbb{E}\left[\#\ \frac{3}{4}\text{-cliques of size } \ell\right] \le \binom{n}{\ell}2e^{-\frac{5}{4}n} \le 2^{n+1}e^{-\frac{5}{4}n} < e^{-\frac{n}{4}}.$$

By Markov's inequality the probability G has a $\frac{3}{4}$-clique of size ℓ is at most $e^{-\frac{n}{4}} \to 0$. This all implies that there exists G with large chromatic number and no $\frac{3}{4}$-clique of size ℓ. If G has a K_m subdivision on $8m$ vertices, then the centres induce a $\frac{3}{4}$-clique of size m. Thus G has no appropriate subdivision. □

Theorem 9 (Weierstrass 1885). The polynomials are dense in the space of continuous functions on $[0, 1]$ with respect to $\|\cdot\|_\infty$.

Proof (Bernstein 1912). Take $f \in \mathcal{C}^0([0, 1], \mathbb{R})$. We may assume $\|f\|_\infty = 1$ by scaling. For $x \in [0, 1]$ let us take $S_n(x) \sim \mathrm{Bin}(n, x)$.

Let $B_n(x) = \mathbb{E}\left[f\left(\frac{S_n(x)}{n}\right)\right] = \sum_{k=0}^{n}\mathbb{P}(S_n(x) = k) f\left(\frac{k}{n}\right) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^k f\left(\frac{k}{n}\right)$. This is a polynomial in x. f is uniformly continuous. Thus for any $\varepsilon > 0$ there exist $\eta > 0$ such that is $|x - y| < \eta$ then $|f(x) - f(y)| < \varepsilon$. This means that

$$|f(x) - B_n(x)| \le \mathbb{E}\left[\left|f(x) - f\left(\frac{S_n(x)}{n}\right)\right|\right] \le$$
$$\varepsilon\mathbb{P}\left(\left|x - \frac{S_n(x)}{n}\right| < \eta\right) + 2\|f\|_\infty \mathbb{P}\left(\left|x - \frac{S_n(x)}{n}\right| > \eta\right).$$

By Chernoff's bound $\mathbb{P}\left(\left|x - \frac{S_n(x)}{n}\right| > \eta\right) \le \varepsilon$ for large enough n. Thus $|f(x) - B_n(x)| \le 3\varepsilon$. □

3. Lovász local lemma

2026-03-16

Idea. Let $A_1, \ldots, A_n$ be a collection of events such that $\mathbb{P}(A_i) < 1$. If they are mutually independent, then $\mathbb{P}\left(\bigwedge_i \overline{A_i}\right) = \prod_{i=1}^{n}\mathbb{P}\left(\overline{A_i}\right) > 0$. Lovász local lemma states something similar for dependent events.

Definition 3. An event A is mutually independent from the set $\{A_f : f \in F\}$ if for any $F' \subseteq F$ A is independent from $\{A_f : f \in F'\}$.

Theorem 10 (Lovász Local Lemma, symmetric version)**.** Let $A_1, \ldots, A_n$ be a collection of events such that for some p and all i we have $\mathbb{P}(A_i) \leq p < 1$ and every A_i is mutually independent from all but d other events. If $4pd \leq 1$ or $ep(d+1) \leq 1$, then $\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.

Proposition 3 (Mutual independence principle)**.** Let $T_1, \ldots, T_m$ be a collection of independent trials such that each A_i is determined by $\{T_j : j \in F_i\}$ for some F_i. Each A_i is mutually independent from $\{A_j : F_i \cap F_j = \emptyset\}$.

Theorem 11. If $\mathcal{H}$ is a hypergraph such that each edge of $\mathcal{H}$ has size at least k and intersects at most $\frac{2^{k-1}}{e} - 1$ edges, then $\mathcal{H}$ is 2-colourable.

Proof. Consider a random uniform 2-colouring of $\mathcal{H}$. Let A_e be the event that e is monochromatic. We have $\mathbb{P}(A_e) \leq 2^{k-1}$. Now every A_e is mutually independent from $\{A_f : e \cap f = \emptyset\}$ – all but $d = \frac{2^{k-1}}{e} - 1$ events. As $ep(d+1) = 1$ by Lovász local lemma there is a colouring of $\mathcal{H}$ that verifies none of $\{A_e\}_{e \in E}$. □

Corollary. For $k \geq 9$ every k-uniform, k-regular hypergraph is 2-colourable.

Proof. Every edge of $\mathcal{H}$ intersects at most $k(k-1)$ other edges. For $k \geq 9$ we have $k(k-1) \leq \frac{2^{k-1}}{e} - 1$. □

Remark. The above corollary does not hold for $k = 2$ (example: C_3) or $k = 3$ (example: Fano plane). Thomassen proved in 1992 that the statement is true for $k \geq 4$.

Theorem 12. Let L be a list assignment of colours to a graph G. If for all $v \in V(G)$ we have $|L(v)| = \ell$ and for all $v \in V(G)$ and $c \in L(v)$ the colour c appears in at most $\frac{\ell}{8}$ lists of neighbours if v, then G is L-colourable.

Proof. Consider a random uniform L-colouring. Let $A_{e,c}$ be the event that both endpoints of e are coloured c. For each e, c we have $\mathbb{P}(A_{e,c}) \leq \frac{1}{\ell^2} := p$.

For each $e = uv$ and every c let $F_u = \{A_{f,c'} : c' \in L(u), u \in f\}$. By mutual independence principle $A_{e,c}$ is mutually independent from $\{A_{f,c'}\} \setminus (F_u \cup F_v)$. We have $|F_u \cup F_v| \leq 2\ell \cdot \frac{\ell}{8} = \frac{\ell^2}{4} := d$.

As $4pd \leq 1$ we conclude by Lovász local lemma. □

Remark. It is conjectured that $\frac{\ell}{8}$ can be replaced with $\ell - 1$. Haxell proved in 2001 that $\frac{\ell}{2}$ works. Reed, Sudakov proved in 2002 that $\ell - o(\ell)$ works.

Theorem 13. If $e\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. Thus

$$\frac{\sqrt{2}}{e}(1 + o(1))\, 2^{\frac{k}{2}} < R(k,k).$$

Proof. Consider a random uniform 2-edge colouring of $G = K_n$. Given a set of vertices S with $|S| = k$ let A_S be the event that $G[S]$ is a monochromatic clique. We have $\mathbb{P}(A_S) = 2^{1-\binom{k}{2}} := p$. A_S is mutually independent from $\{A_T : |T \cap S| \leq 1\}$. We have

$$|\{A_T : |T \cap S| \geq 2\}| = \binom{k}{2}\binom{n-2}{k-2} - 1 := d.$$

As $epd \leq 1$ by Lovász local lemma we have $\mathbb{P}\left(\bigwedge_{|S|=k} \overline{A_S}\right) > 0$ and K_n has a 2-edge colouring without a monochromatic K_k. □

Remark. The usefulness of Lovász local lemma is limited here, as d is large. For off-diagonal Ramsey numbers it gives a lot better bounds.

Theorem 14 (Lovász Local Lemma, asymmetric version)**.** Let $A_1 \dots, A_n$ be a collection of events. Let D be the dependency graph on $\{A_i\}_{i\in[n]}$. Assume that A_i is mutually independent from $\{A_j : ij \notin D\}$ for any i and there exist $x_1, \dots, x_n \in [0,1]$ such that $\mathbb{P}(A_i) \leq x_i \prod_{ij\in E(D)} (1-x_j)$ for all i. Then $\mathbb{P}\left(\bigwedge_{i=1}^n \overline{A_i}\right) \geq \prod_{i=1}^n (1-x_i)$.

Proof. We first prove by induction on s that for every set $S \subseteq \{1, \dots, n\}$ with $|S| = s$ and for every i we have $\mathbb{P}\left(A_i \mid \bigwedge_{j\in S} \overline{A_j}\right) \leq x_i$. $s = 0$ follows by one of the assumptions. Assume the statement holds for every $s' < s$. Let $S_1 = \{j \in S : ij \in E(D)\}$ and $S_2 = S \setminus S_1$. We have

$$\mathbb{P}\left(A_i \mid \bigwedge_{j\in S} \overline{A_j}\right) = \frac{\mathbb{P}\left(A_i \wedge \left(\bigwedge_{j\in S_1} \overline{A_j}\right) \mid \bigwedge_{j\in S_2} \overline{A_j}\right)}{\mathbb{P}\left(\bigwedge_{j\in S_1} \overline{A_j} \mid \bigwedge_{j\in S_2} \overline{A_j}\right)} \leq \frac{\mathbb{P}\left(A_i \mid \bigwedge_{j\in S_2} \overline{A_j}\right)}{\mathbb{P}\left(\bigwedge_{j\in S_1} \overline{A_j} \mid \bigwedge_{j\in S_2} \overline{A_j}\right)} =$$
$$\frac{\mathbb{P}(A_i)}{\mathbb{P}\left(\bigwedge_{j\in S_1} \overline{A_j} \mid \bigwedge_{j\in S_2} \overline{A_j}\right)} \leq \frac{x_i \prod_{ij\in E(D)} (1-x_j)}{\mathbb{P}\left(\bigwedge_{j\in S_1} \overline{A_j} \mid \bigwedge_{j\in S_2} \overline{A_j}\right)}.$$

The second equality follows from A_i being mutually independent from S_2. Denote $S_1 = \{j_1, \dots, j_r\}$. If $r = 0$, the denominator is 1 and the statement holds. Otherwise

$$\mathbb{P}\left(\overline{A_{j_1}} \wedge \dots \wedge \overline{A_{j_r}} \mid \bigwedge_{e\in S_2} \overline{A_e}\right) =$$
$$\left(1 - \mathbb{P}\left(A_{j_1} \mid \bigwedge_{e\in S_2} \overline{A_e}\right)\right) \dots \left(1 - \mathbb{P}\left(A_{j_r} \mid \overline{A_{j_1}} \wedge \dots \wedge \overline{A_{r-1}} \wedge \bigwedge_{e\in S_2} \overline{A_e}\right)\right) \geq$$
$$(1-x_{j_1}) \dots (1-x_{j_r}) = \prod_{ij\in E(D)\cap S} (1-x_j) \geq \prod_{ij\in E(D)} (1-x_j).$$

Thus the inductive statement holds. We conclude by computing

$$\mathbb{P}\left(\bigwedge_{i=1}^n \overline{A_i}\right) = (1 - \mathbb{P}(A_1)) \dots \left(1 - \mathbb{P}\left(A_n \mid \overline{A_1} \wedge \dots \wedge \overline{A_{n-1}}\right)\right) \geq \prod_{i=1}^n (1-x_i).$$

□

Remark. The symmetric case follows from this statement.

Definition 4. Let Σ be an alphabet. A non-repetitive (a separator-free) word w on Σ is a word using the letters of Σ and containing no subword of the form xx.

Theorem 15 (Thue, 1906)**.** For every $n \geq 4$ there is a non-repetitive word of length n on Σ if $|\Sigma| \geq 3$.

Definition 5. Let $\mathrm{sf}(n)$ be the smallest k such that for every sequence $\Sigma_1, \dots, \Sigma_n$ of alphabets of size k, there exists a square-free word w on $\Sigma_1 \dots \Sigma_n$.

Theorem 16. $\mathrm{sf}(n) < +\infty$.

Proof. Let $\Sigma_1, \dots, \Sigma_n$ be alphabets of size $k = 16$. Let w be a random, uniform word on $\Sigma_1 \dots \Sigma_n$. Let $B_{i,\ell}$ be the event that $w_i \dots w_{i+2\ell-1}$ is a square. We have $\mathbb{P}(B_{i,\ell}) \leq k^{-\ell}$. $B_{i,\ell}$ is mutually independent from all events but $D_{i,\ell} = \{B_{j,m} : j \in [i-2m+1, i+2\ell-1]\}$ (the subwords that overlap).

We define $x_{i,\ell} = x_\ell = \frac{c^\ell}{k^\ell + c^\ell}$ for some c. We have

$$x_{i,\ell} \prod_{D_{i,\ell}} (1 - x_{j,m}) = x_\ell \prod_{m\geq 1} (1-x_m)^{2(m+\ell-1)} = \frac{c^\ell}{k^\ell + c^\ell} \prod_{m\geq 1} \left(\frac{k^m}{k^m + c^m}\right)^{2(\ell+m-1)}.$$

Using $\frac{k^m}{k^m+c^m} \geq \exp\left(-\frac{c^m}{k^m}\right)$ and setting $\delta^m = \frac{c^m}{k^m}$ we have that this is not smaller than

$$\frac{c^\ell}{k^\ell + c^\ell} \exp\left(-\sum_{m\geq 1} 2\left(\ell+m-1\right)\delta^m\right) \geq$$
$$\frac{c^\ell}{k^\ell + c^\ell} \exp\left(-2\left(\left(\ell-2\right)\sum_{m\geq 1}\delta^m + \sum_{m\geq 1}\left(m+1\right)\delta^m\right)\right) \geq$$
$$\exp\left(-\frac{2\ell}{7}+1\right)\frac{c^\ell}{k^\ell + c^\ell}.$$

The last inequality follows by some calculus. Setting $c = 2$ and using the asymmetric Lovász local lemma we conclude. □

4. Random graphs and thresholds

2026-03-23

Theorem 17. $G\left(n,p\right)$ contains a triangle with probability $o\left(1\right)$ if $p \ll \frac{1}{n}$ and with probability $1-o\left(1\right)$ if $p \gg \frac{1}{n}$.

Proof. Assume $p \ll \frac{1}{n}$. Let X be the number of triangles in $G\left(n,p\right)$. We have $\mathbb{E}\left[X\right] = \binom{n}{3}p^3 \leq n^3p^3 = o\left(1\right)$. We conclude by Markov's inequality.

Now assume $p \gg \frac{1}{n}$. Let X_{uvw} be the indicator of a triangle on uvw. Let X be the sum of X_{uvw} for all triples of vertices. After come calculations we get $\operatorname{Var}\left(X\right) = O\left(n^3\right)\left(p^3-p^2\right) + O\left(n^4\right)\left(p^6-p^6\right) = o\left(n^6p^6\right) \sim \mathbb{E}\left[X\right]^2$. Thus we have

$$\mathbb{P}\left(X < \left(1-\lambda\right)\mathbb{E}\left[X\right]\right) \leq \frac{\operatorname{Var}\left(X\right)}{\lambda^2\mathbb{E}\left[X\right]^2} = \frac{1}{\lambda^2}o\left(1\right) = o\left(1\right).$$

□

Theorem 18 (Poisson limit). If $np \to 0$, then $\mathbb{P}\left(G\left(n,p\right)\text{ contains a triangle}\right)$ approaches the Poisson distribution.

Theorem 19 (Asymptotic normality).

$$\frac{X_n - \mathbb{E}\left[X_n\right]}{\sqrt{\operatorname{Var}\left(X_n\right)}} \to \mathcal{N}\left(0,1\right).$$

Lemma 1. Let X_n be such that $\mathbb{E}\left[X_n\right] \to +\infty$. If $\operatorname{Var}\left(X_n\right) = o\left(\mathbb{E}\left[X_n\right]^2\right)$ then $X_n \sim \mathbb{E}\left[X_n\right]$ (asymptotic equality).

Proof. By Chebyshev's inequality $\mathbb{P}\left(\left|X-\mathbb{E}\left[X\right]\right| > \lambda\mathbb{E}\left[X\right]\right) \leq \frac{\operatorname{Var}(X)}{\lambda^2\mathbb{E}[X]^2} = o\left(1\right)$. □

Lemma 2. Let $X = \sum X_i$, where X_i are symmetric indicators with $\mathbb{E}\left[X\right] \to +\infty$. Let $I^* = \{(i,j) : i < j \text{ and } X_i, X_j \text{ are not independent}\}$, $\Delta = \sum_{I^*}\mathbb{P}\left(A_i \cap A_j\right)$, where $A_i = \{X_i = 1\}$. If $\Delta = o\left(\mathbb{E}\left[X\right]^2\right)$, then $X \sim \mathbb{E}\left[X\right]$ with high probability.

Proof. We have $\operatorname{Var}\left(X\right) = \sum_i \operatorname{Var}\left(X_i\right) + \sum_{(i,j)\in I^*}\operatorname{Cov}\left(X_i,X_j\right)$ and $\operatorname{Cov}\left(X_i,X_j\right) = \mathbb{E}\left[X_iX_j\right] - \mathbb{E}\left[X_i\right]\mathbb{E}\left[X_j\right] \leq \mathbb{E}\left[X_iX_j\right] = \mathbb{P}\left(A_i \cap A_j\right)$. Thus

$$\operatorname{Var}\left(X\right) \leq \sum_i \mathbb{E}\left[X_i\right] + \sum_{I^*}\mathbb{P}\left(A_i \cap A_j\right) = \mathbb{E}\left[X\right] + \Delta.$$

We conclude by the previous lemma. □

Lemma 3. Let $X = \sum X_i$, where X_i are symmetric indicators with $\mathbb{E}[X] \to +\infty$. Let $I^* = \{(i,j) : i < j \text{ and } X_i, X_j \text{ are not independent}\}$, $\Delta^* = \sum_{(i,j)\in I^*} \mathbb{P}(A_j \mid A_i)$, where $A_i = \{X_i = 1\}$. If $\Delta^* = o(\mathbb{E}[X])$, then $X \sim \mathbb{E}[X]$ with high probability.

Proof.

$$\Delta = \sum_{I^*} \mathbb{P}(A_i \cap A_j) = \sum_i \mathbb{P}(A_i) \cdot \sum_{(i,j)\in I^*} \mathbb{P}(A_j \mid A_i).$$

We set $\Delta^* = \sum_{(i,j)\in I^*} \mathbb{P}(A_j \mid A_i)$ for some fixed i (by symmetry the value is independent of i). We get $\Delta = \mathbb{E}[X]\Delta^*$ and conclude by the previous lemma. □

Definition 6. We say a graph property is monotone if adding edges does not destroy the property.

A monotone property P has threshold $\nu(n)$ if $\mathbb{P}(G(n,p) \text{ verifies } P) \to \begin{cases} 0, & p \ll \nu(n) \\ 1, & p \gg \nu(n) \end{cases}$.

Theorem 20 (Bollobás, Thomassen 1987)**.** Every non-trivial monotone graph property has a threshold.

Definition 7. We say a property P has a sharp threshold $\nu(n)$ if for every $\delta > 0$

$$\mathbb{P}(G(n,p) \text{ verifies } P) \to \begin{cases} 0, & \frac{p}{\nu(n)} \le 1-\delta \\ 1, & \frac{p}{\nu(n)} \ge 1+\delta \end{cases}.$$

Definition 8. We say a property P has a coarse threshold $\nu(n)$ if there exist $\varepsilon > 0$ and $0 < c < C$ such that $\mathbb{P}(G(n,p) \text{ verifies } P) \in [\varepsilon, 1-\varepsilon]$ if $c \le \frac{p}{\nu(n)} \le C$.

Theorem 21 (Friedgut 1999)**.** All monotone graph properties with a coarse threshold can be approximated by a local property.

Definition 9. We define the maximal average degree of H as $m(H) = \max_{H'\subseteq H} \frac{|E(H')|}{|V(H')|}$.

Theorem 22 (Goldberg 1984)**.** $m(H)$ can be computed in polynomial time.

Theorem 23 (Bollobás 1981)**.** The threshold for containing H as a subgraph is coarse and of order $n^{-\frac{1}{m(H)}}$.

Theorem 24. Let $p = \frac{\ln n + c_n}{n}$.

$$\mathbb{P}(G(n,p) \text{ is connected}) \to \begin{cases} 0, & c_n \to -\infty \\ 1 - e^{e^{-c}}, & c_n \to c \\ 1, & c_n \to +\infty \end{cases}.$$

Theorem 25. Let X_k count the number of cliques of size k in $G\left(n, \frac{1}{2}\right)$. If $k \ge (2+\delta)\log_2 n$, then $\omega\left(G\left(n,\frac{1}{2}\right)\right) < k$ with high probability. If $k \le (2-\delta)\log_2 n$, then $\omega\left(G\left(n,\frac{1}{2}\right)\right) \ge k$ with high probability.

Proof. $\mathbb{E}[X_k] = \binom{n}{k} 2^{-\binom{k}{2}} =: f(n,k)$. We have $\left(\frac{n}{ek}\right)^k \le \binom{n}{k} \le \left(\frac{ne}{k}\right)^k$ by Stirling approximation. Thus $\log_2 f(n,k) = k\left(\log_2 n - \log_2 k - \frac{k}{2} + o(1)\right)$. We have $f(n,k) \to 0$ for $k \ge (2+\delta)\log_2 n$ and $f(n,k) \to +\infty$ for $k \le (2-\delta)\log_2 n$. The statement for large k follows by first moment method.

Assume $k \le (2-\delta)\log_2 n$. Given a set S of k vertices let $A_S \equiv S$ induces a clique. We have $\Delta^* = \sum_{T:|S\cap T|\ge 2} \mathbb{P}(A_T \mid A_S) = \sum_{i=2}^{k-1} \binom{k}{i}\binom{n-k}{k-i} 2^{\binom{i}{2}-\binom{k}{2}} = o(\mathbb{E}[X])$. The statement follows from one of the lemmas. □

Theorem 26 (Bollobás, Erdős 1976; Matula 1976)**.** There exists $k(n)$ with $k(n) \sim 2\log_2 n$ such that with high probability $\omega\left(G\left(n, \frac{1}{2}\right)\right) \in \{k(n), k(n)+1\}$.

Proof. For $k \sim 2\log_2 n$ we have

$$\frac{f(n,k+1)}{f(n,k)} = \frac{\binom{n}{k+1} 2^{-\frac{k(k+1)}{2}}}{\binom{n}{k} 2^{-\frac{k(k-1)}{2}}} = \frac{n-k}{k-1} 2^{-k} = n^{-1+o(1)}.$$

Let k_0 be the value such that $f(n,k_0) \geq n^{-\frac{1}{2}} > f(n,k_0+1)$. Then $f(n,k_0-1) \to +\infty$ and $f(n,k_0+1) = o(1)$. This means $\omega\left(G\left(n,\frac{1}{2}\right)\right) \in \{k_0-1, k_0\}$ with high probability. □